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Laguerre–Freud's Equations for the Recurrence Coefficients of Semi-classical Orthogonal Polynomials

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In this paper we give a new scheme for deriving a non-linear system, satisfied by the three-term recurrence coefficients of semi-classical orthogonal Polynomials, this non-linear system is labelled "Laguerre–Freud's equations." Here we do not deal with the numerical aspect of the question (stability, asymptotic, ...). Our purpose is to take due advantage of linear functionals formalism and to show that given a semi-classical linear functional, i.e., given two polynomials, we are able to provide the Laguerre–Freud's equations. The way of obtaining these equations is put in a recursive form appropriate for computer algebra calculation, especially when the degrees of the two given polynomials are large. We illustrate our process by several examples and we point out two cases where the solutions to Laguerre–Freud's equations are not unique.

I. INTRODUCTION

Let $\{P_n\}_{n\geqslant 0}$ be a sequence of monic Orthogonal Polynomials (OP) with respect to the linear functional $\mathscr{L}$, and therefore $\{P_n\}_{n\geqslant 0}$ satisfies a three-term recurrence relation, i.e., [9],

$$\begin{aligned} P_{n+2}(x) &= (x-\beta_{n+1})\,P_{n+1}(x)-\gamma_{n+1}\,P_n(x) \qquad n\geqslant 0 \\ P_1(x) &= x-\beta_0, \qquad\qquad P_0(x)=1 \end{aligned} \tag{1}$$

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where $\beta_n, \gamma_{n+1} \in \mathbb{C}$, $\gamma_{n+1} \neq 0$, $n \geqslant 0$ β_n and γ_n are given by

$$\beta_n = \frac{\langle \mathscr{L}, xP_n^2(x) \rangle}{\langle \mathscr{L}, P_n^2 \rangle}, \qquad \gamma_{n+1} = \frac{\langle \mathscr{L}, P_{n+1}^2 \rangle}{\langle \mathscr{L}, P_n^2 \rangle}, \qquad n \geqslant 0 \tag{2}$$

with the convention $\gamma_0 = \langle \mathscr{L}, P_0^2 \rangle$, in order that $\langle \mathscr{L}, P_n^2 \rangle = \prod_{j=0}^{n} \gamma_j$ for all $n \geqslant 0$.

In numerous applications [6, 11, 13, 12, 28] one needs information about the zeros of $\{P_n\}_{n \geqslant 0}$. The usual way of tackling this question is to use Sturm-type analysis applied to the differential equation satisfied by the corresponding polynomials. However, orthogonal polynomials generated by the three-term recurrence relation do not necessarily satisfy any simple differential equation, and if they do [4, 27] then the nature of the differential equation is not always appropriate for Sturm-type analysis. Thus the suitable approach is to treat the zeros of orthogonal polynomials as the eigenvalues of truncated Jacobi matrices; according to this point of view the recurrence coefficients β_n and γ_n are essential ingredients.

Here we present a straightforward way which allows us to establish the non-linear system satisfied by β_n and γ_n, requiring only the assumption that $\mathscr{L}$ is semi-classical [15, 24] i.e.,

$$\psi \mathscr{L} + D[\phi \mathscr{L}] = 0, \tag{3}$$

where

$$\langle \psi \mathscr{L}, P \rangle := \langle \mathscr{L}, \psi P \rangle$$

$$\langle D[\phi \mathscr{L}], P \rangle := -\langle \phi \mathscr{L}, D[P] \rangle$$

$$= -\langle \phi \mathscr{L}, P' \rangle$$

$P := P(x)$ an arbitrary polynomial, ϕ and ψ polynomials in x

The integer $s = \max\{\text{degree } \psi - 1, \text{degree } \phi - 2\}$ is the *class* of $\mathscr{L}$. If $\mathscr{L}$ is represented by a positive weight function, say ρ over the interval I, i.e.,

$$\langle \mathscr{L}, P \rangle := \int_I \rho(x) \, P(x) \, dx,$$

then (3) is equivalent to [3],

$$(\psi + \phi')\rho + \phi \rho' = 0$$

and $\phi(x) \, \rho(x) \, P(x)|_I = 0$ for any arbitrary polynomial P.

To recover Hendriksen and van Rossum notation [15], just set

$$A = \psi + \phi'$$
$$B = \phi.$$

The non-linear system will be labelled Laguerre–Freud's equations, we borrowed this denomination from A. Magnus [22, 23], who dealt with the problem in the case of an exponential weight. We will see later that our approach, with the suitable substitutions of the parameters, agrees with Magnus' results.

A number of problems in physical science [1, 8, 10, 16–18, 21, 29–31] and in applied mathematics deal with semi-classical polynomials and particularly with the class $s = 1$ [2, 3]. As an illustration of our process we write the explicit Laguerre–Freud's equations in the case where $\mathscr{L}$ is semi-classical of class one, as well as when $\mathscr{L}$ is represented by the weight function $\rho(x) = \exp\{-Q(x)\}$ over the real line, with

$$Q(x) = \sum_{k=0}^{2m} q_k x^k, \qquad q_{2m} > 0$$

At the end, we show on an example how the complexity of the non-linear system increases by writing the Laguerre–Freud's equations for

$$\rho_1(x) = e^{-x^2} \quad \text{on} \quad]-\infty, +\infty[\quad \text{(Hermite OP)} \tag{4}$$
$$\rho_2(x) = e^{-x^2} \quad \text{on} \quad [0, +\infty[\quad \text{(Maxwell OP)} \tag{5}$$
$$\rho_3(x) = e^{-x^2} \quad \text{on} \quad [0, 1] \quad \text{(Truncated Maxwell OP)} \tag{6}$$
$$\tilde{\rho}_3(x) = e^{-x^2} \quad \text{on} \quad]-\infty, 0]\, U[1, +\infty[\tag{7}$$
$$\rho_4(x) = e^{-x^{2m}} \quad \text{on} \quad [-c, c] \tag{8}$$
$$\tilde{\rho}_4(x) = e^{-x^{2m}} \quad \text{on} \quad]-\infty, -c]\, U[c, +\infty[\tag{9}$$

which satisfy, respectively,

$$2x\mathscr{L}_1 + D[\mathscr{L}_1] = 0 \tag{10}$$
$$(2x^2 - 1)\mathscr{L}_2 + D[x\mathscr{L}_2] = 0 \tag{11}$$
$$\begin{cases} (2x^3 - 2x^2 - 2x + 1)\mathscr{L}_3 + D[x(x-1)\mathscr{L}_3] = 0 & (12) \\ (2x^3 - 2x^2 - 2x + 1)\tilde{\mathscr{L}}_3 + D[x(x-1)\tilde{\mathscr{L}}_3] = 0 & (13) \end{cases}$$
$$\begin{cases} 2(mx^{2m+1} - mc^2x^{2m-1} - x)\mathscr{L}_4 + D[(x^2 - c^2)\mathscr{L}_4] = 0 & (14) \\ 2(mx^{2m+1} - mc^2x^{2m-1} - x)\tilde{\mathscr{L}}_4 + D[(x^2 - c^2)\tilde{\mathscr{L}}_4] = 0. & (15) \end{cases}$$

The last four linear functionals exemplify the situation where the Laguerre–Freud's equations have more than one solution.

Equations (10)–(15) are obtained via the recurrence relation satisfied by the moments, for instance, let us take the Maxwell case:

$$\mu_n = \int_0^{+\infty} x^n e^{-x^2}\, dx,$$

the moments satisfy

$$2\mu_{n+2} - (n+1)\mu_n = 0, \qquad n \geqslant 0,$$

which may be written as

$$(2\mu_{n+2} - \mu_n) - n\mu_n = 0, \qquad n \geqslant 0,$$

$$2\langle \mathscr{L}_2, x^{n+2} \rangle - \langle \mathscr{L}_2, x^n \rangle - n\langle \mathscr{L}_2, x^n \rangle = 0, \qquad n \geqslant 0$$

$$\langle (2x^2-1)\mathscr{L}_2, x^n \rangle - n\langle x\mathscr{L}_2, x^{n-1} \rangle = 0, \qquad n \geqslant 0$$

$$\langle (2x^2-1)\mathscr{L}_2, x^n \rangle - \langle x\mathscr{L}_2, (x^n)' \rangle = 0, \qquad n \geqslant 0.$$

Finally we obtain

$$\langle (2x^2-1)\mathscr{L}_2 + D[x\mathscr{L}_2], x^n \rangle 0, \qquad n \geqslant 0$$

hence Eq. (11).

As we said in the abstract, here we do not deal with the numerical aspect of Laguerre–Freud's equations. We conclude this introduction with a few words about the efficiency of Laguerre–Freud's equations. In principle the first $2n$ moment $\mu_k = \langle \mathscr{L}, x^k \rangle$, $k = 0, 1, ..., 2n-1$ determine uniquely the $2n$ recurrence coefficients β_k and γ_k via Eq. (2), for this we have to perform cn operations (c constant) to evaluate the numerator and the denominator of the fractions in (2). Further we need the intermediate polynomials; that is to say that we have to perform $c(1+2+\cdots+n) \sim cn^2/2$ operations to reach the level n. The modified Chebyshev algorithm requires $O(n^2)$ operations [13], but Laguerre–Freud's equations request $O(n)$ operations for the computation of β's and γ's of level n.

II. Preliminary Results

In the sequel we need the following results.

A. *Iteration of the Three-Term Recurrence Relation*

If we start with the three-term recurrence relation and we repeat the process we obtain

$$x^k P_n(x) = \sum_{j=n-k}^{n+k} C_{j,n}^k P_j(x), \tag{16}$$

with the convention that polynomials with negative subscript are zero. It is easy to see that

LEMMA 2.1. *The coefficients* $C^k_{j,n}$ *satisfy*

$$C^{k+1}_{n-k-1,n}=\gamma_{n-k}C^k_{n-k,n}=\prod_{j=0}^{k}\gamma_{n-j}\qquad (C^k_{j,n}\equiv 0 \text{ for } j<0)$$

$$C^{k+1}_{n-k,n}=\beta_{n-k}C^k_{n-k,n}+\gamma_{n-k+1}C^k_{n-k+1,n}$$

$$C^{k+1}_{j,n}=C^k_{j-1,n}+\beta_j C^k_{j,n}+\gamma_{j+1}C^k_{j+1,n},\qquad n-k+1\leqslant j\leqslant n+k-1$$

$$C^{k+1}_{n+k,n}=C^k_{n+k-1,n}+\beta_{n+k}C^k_{n+k,n}$$

$$C^{k+1}_{n+k+1,n}=C^k_{n+k,n}=1.$$

Starting with the vector $\mathbf{V}_1=(C^1_{n-1,n}, C^1_{n,n}, C^1_{n+1,n})^T$ a matrix version of the above relations may be written, which permits the evaluation of the vector

$$\mathbf{V}_{k+1}=(C^{k+1}_{n-k-1,n}; C^{k+1}_{n-k,n}; \ldots C^{k+1}_{j,n}; \ldots C^{k+1}_{n+k+1,n})^T$$

as

$$\mathbf{V}_{k+1}=\prod_{j=0}^{k-1}A_{k-j}\mathbf{V}_1 \tag{17}$$

$$C^1_{n-1,n}=\gamma_n\quad (n\geqslant 1),\qquad C^1_{n,n}=\beta_n\quad (n\geqslant 0),\qquad C^1_{n+1,n}=1\quad (n\geqslant 0),$$

where the $(2k+3)\times(2k+1)$ matrix A_k is given by:

$$A_k=\begin{bmatrix} \gamma_{n-k} & 0 & \cdots & 0 & 0 \\ \beta_{n-k} & \gamma_{n-k+1} & & & \cdot \\ 1 & \beta_{n-k+1} & & & \cdot \\ \cdot & 1 & & & \cdot \\ \cdot & \cdot & & & 0 \\ & \cdot & 1 & \beta_{n+k-1} & \gamma_{n+k} \\ 0 & 0 & & 1 & \beta_{n+k} \\ 0 & 0 & & 0 & 1 \end{bmatrix}. \tag{18}$$

This process, is of course equivalent to a more familiar one involving the tridiagonal Jacobi matrix associated to the recurrence relation (1) [22].

B. *Turan Determinant*

Let $\mathcal{T}_{n+1}=P_nP_{n+2}-P^2_{n+1}$ denote the Turán determinant, which satisfies the recurrence

$$\mathcal{T}_{n+1}-\gamma_n\mathcal{T}_n=(\beta_n-\beta_{n+1})P_{n+1}P_n+(\gamma_n-\gamma_{n+1})P^2_n\qquad (n\geqslant 1). \tag{19}$$

It is an easy task to show

LEMMA 2.2. *The Turán determinants are given by*

$$\mathcal{T}_{n+1} = -\prod_{j=0}^{n} \gamma_j + \sum_{h=0}^{n} \prod_{j=0}^{n-1-h} \gamma_{n-j} F(h), \qquad n \geqslant 0,$$

where

$$F(h) = (\beta_h - \beta_{h+1})\, P_{h+1} P_h + (\gamma_h - \gamma_{h+1}) P_h^2.$$

In the sequel we will assume that empty products are equal to unity and empty sums are equal to zero.

III. Laguerre–Freud's Equations

A. *Algorithm*

Let us start with the functional equation for the semi-classical form $\mathcal{L}$ (Eq. (3)), where the polynomials $\phi(x)$ and $\psi(x)$ are written as

$$\phi(x) = \sum_{i=0}^{t} c_i x^i, \qquad t \geqslant 0, \tag{20}$$

$$\psi(x) = \sum_{j=0}^{p} a_j x^j, \qquad p \geqslant 1. \tag{21}$$

The null form given in Eq. (3) acting on P_n^2 and $P_{n+1}P_n$ gives the two relations,

$$\langle \psi\mathcal{L}, P_n^2 \rangle = 2 \langle \phi\mathcal{L}, P_n P_n' \rangle \tag{22}$$

$$\langle \psi\mathcal{L}, P_{n+1} P_n \rangle = \langle \phi\mathcal{L}, (P_n P_{n+1})' \rangle. \tag{23}$$

Equations (22) and (23) will provide the *Laguerre–Freud's equations for the linear functional* $\mathcal{L}$. After expanding ϕ and ψ according to (20) and (21), the last identities become

$$\begin{cases} \displaystyle\sum_{j=0}^{p} a_j \langle x^j \mathcal{L}, P_n^2 \rangle = 2 \sum_{i=0}^{t} c_i \langle x^i \mathcal{L}, P_n P_n' \rangle & (24) \\ \displaystyle\sum_{j=0}^{p} a_j \langle x^j \mathcal{L}, P_{n+1} P_n \rangle = \sum_{i=0}^{t} c_i \langle x^i \mathcal{L}, (P_n P_{n+1})' \rangle. & (25) \end{cases}$$

The recurrence coefficients β_n and γ_n already contained in these 2 relations replacing, in a clever way, xP_n by $P_{n+1}+\beta_n P_n+\gamma_n P_{n-1}$ and evaluating the four "integrals":

$$I_{k,n} := \langle x^k \mathscr{L}, P_n^2 \rangle, \qquad I_{0,n} = \prod_{j=0}^{n} \gamma_j \tag{26}$$

$$J_{k,n} := \langle x^k \mathscr{L}, P_n P_n' \rangle, \qquad J_{0,n} = 0, \qquad J_{1,n} = nI_{0,n} \tag{27}$$

$$K_{k,n} := \langle x^k \mathscr{L}, P_{n+1} P_n \rangle, \qquad K_{0,n} = 0, \qquad K_{1,n} = \gamma_{n+1} I_{0,n} \tag{28}$$

$$L_{k,n} := \langle x^k \mathscr{L}, (P_{n+1} P_n)' \rangle, \qquad L_{0,n} = (n+1) I_{0,n}. \tag{29}$$

Using the orthogonality of the sequence $\{P_n\}_{n\geqslant 0}$ with respect to $\mathscr{L}$ and Lemma 2.1, we have

LEMMA 3.1. *The "integrals" $I_{k,n}$ and $K_{k,n}$ are given by*

$$I_{k,n} = C_{n,n}^k I_{0,n}$$
$$K_{k,n} = C_{n+1,n}^k I_{0,n+1} = \gamma_{n+1} C_{n+1,n}^k I_{0,n}.$$

Again, using the orthogonality of $\{P_n\}_{n\geqslant 0}$ with respect to $\mathscr{L}$, we obtain

LEMMA 3.2. *The "integrals" $J_{k,n}$ and $L_{k,n}$ satisfy mixed recurrences in k for a fixed n,*

$$2J_{k+1,n} = 2\beta_n J_{k,n} + L_{k,n} + \gamma_n L_{k,n-1} - I_{k,n}$$
$$L_{k+1,n} = 2\gamma_{n+1} J_{k,n} + \beta_{n+1} L_{k,n} - K_{k,n} + \langle x^k \mathscr{L}, (P_n P_{n+2})' \rangle.$$

The evaluation of $< x^k \mathscr{L}, (P_n P_{n+2})' \rangle$ is controlled via the Turán determinant; indeed we have

$$(P_{n+2} P_n)' = \mathscr{T}'_{n+1} + 2P_{n+1} P'_{n+1}. \tag{30}$$

Thus

$$\langle x^k \mathscr{L}, (P_n P_{n+2})' \rangle = \langle x^k \mathscr{L}, \mathscr{T}'_{n+1} \rangle + 2J_{k,n+1}. \tag{31}$$

According to Lemma 2.2 we have

$$\langle x^k \mathscr{L}, \mathscr{T}'_{n+1} \rangle = \sum_{h=0}^{n} \prod_{j=0}^{n-1-h} \gamma_{n-j} \{ (\beta_h - \beta_{h+1}) \langle x^k \mathscr{L}, (P_h P_{h+1})' \rangle$$
$$+ (\gamma_h - \gamma_{h+1}) \langle x^k \mathscr{L}, (P_h^2)' > \}$$

or

$$\langle x^k \mathscr{L}, \mathscr{T}'_{n+1} \rangle = \sum_{h=0}^{n} \prod_{j=0}^{n-1-h} \gamma_{n-j} \{ (\beta_h - \beta_{h+1}) L_{k,h} + 2(\gamma_h - \gamma_{h+1}) J_{k,h} \}.$$

We may translate the Lemma 3.2 to a matrix version, namely,

$$\begin{bmatrix} 2J_{k+1,n} \\ L_{k+1,n} \end{bmatrix} = \begin{bmatrix} \beta_n & 1 \\ \gamma_{n+1} & \beta_{n+1} \end{bmatrix} \begin{bmatrix} 2J_{k,n} \\ L_{k,n} \end{bmatrix} + \begin{bmatrix} \gamma_n L_{k,n-1} - I_{k,n} \\ 2J_{k,n+1} - K_{k,n} + \langle x^k \mathscr{L}, \mathscr{T}'_{n+1} \rangle \end{bmatrix}. \tag{32}$$

With obvious notation we have

$$\mathbf{Y}_{k+1,n} = B_n \mathbf{Y}_{k,n} + \mathbf{Z}_{k,n}.$$

In other words

$$\mathbf{Y}_{k+1,n} = (B_n)^k \mathbf{Y}_{1,n} + \sum_{j=0}^{k-1} (B_n)^j \mathbf{Z}_{k-j,n}.$$

According to Lemma 2.1, the results of Lemmas 3.1 and 3.2 mean that if we know the values of the four "integrals" at level k, we are able to compute them at level $k+1$. As stated by (24) and (25) we need only to perform the recurrences satisfied by I, J, K, L for $k \leqslant \max\{p, t\}$. At this stage we have all the ingredients involved in Eqs. (24) and (25).

B. *Laguerre–Freud's Equations for Semi-Classical Orthogonal Polynomials of Class One*

Let us write the "integrals" which are involved in the case where $\mathscr{L}$ is semi-classical of class one [2, 3], i.e.,

$$\psi(x) = a_2 x^2 + a_1 x + a_0, \qquad \phi(x) = c_3 x^3 + c_2 x^2 + c_1 x + c_0.$$

According to Eqs. (24) and (25) we need

$$I_{k,n}, K_{k,n}, 0 \leqslant k \leqslant 2, \qquad n \geqslant 0$$
$$J_{k,n}, L_{k,n}, 0 \leqslant k \leqslant 3, \qquad n \geqslant 0$$

which are computed via the previous algorithm

$$\begin{cases} I_{0,n} = \langle \mathscr{L}, P_n^2 \rangle \\ I_{1,n} = \beta_n I_{0,n} \\ I_{2,n} = (\gamma_{+1} + \gamma_n + \beta_n^2) I_{0,n}, \qquad n \geqslant 1 \\ I_{2,0} = (\gamma_1 + \beta_0^2) I_{0,0} \end{cases} \tag{33}$$

$$\left\{\begin{aligned} J_{0,n} &= \langle \mathscr{L}, P_n P'_n \rangle = 0 \\ J_{1,n} &= n I_{0,n} \\ J_{2,n} &= \left[(n-1)\beta_n + \sum_{k=0}^{n} \beta_k \right] I_{0,n}, \qquad n \geqslant 0 \\ J_{3,n} &= \left\{ n(\gamma_{n+1} + \gamma_n) + \beta_n \left[(n-1)\beta_n + \sum_{k=0}^{n} \beta_k \right] \right. \\ &\quad \left. + 2 \sum_{k=1}^{n-1} \gamma_k + \sum_{k=0}^{n-1} \beta_k^2 \right\} I_{0,n} n \geqslant 0 \end{aligned}\right. \tag{34}$$

$$\left\{\begin{aligned} K_{0,n} &= \langle \mathscr{L}, P_{n+1} P_n \rangle = 0 \\ K_{1,n} &= \gamma_{n+1} I_{0,n}, \qquad n \geqslant 0 \\ K_{2,n} &= \gamma_{n+1}(\beta_{n+1} + \beta_n) I_{0,n}, \qquad n \geqslant 0 \end{aligned}\right. \tag{35}$$

$$\left\{\begin{aligned} L_{0,n} &= (n+1) I_{0,n} \\ L_{1,n} &= \left(\sum_{k=0}^{n} \beta_k \right) I_{0,n} \\ L_{2,n} &= \left[(2n+1)\gamma_{n+1} + 2 \sum_{k=1}^{n} \gamma_k + \sum_{k=0}^{n} \beta_k^2 \right] I_{0,n}, \qquad n \geqslant 0 \\ L_{3,n} &= \left\{ \gamma_{n+1} [(2n+1)\beta_{n+1} + 2n\beta_n] + 3 \sum_{k=1}^{n} \gamma_k (\beta_k + \beta_{k-1}) + \sum_{k=0}^{n} \beta_k^3 \right. \\ &\quad \left. + 2\gamma_{n+1} \sum_{k=0}^{n} \beta_k \right\} I_{0,n}, \qquad n \geqslant 0. \end{aligned}\right. \tag{36}$$

Equations (24) and (25) can be written respectively as

$$a_2 I_{2,n} + a_1 I_{1,n} + a_0 I_{0,n} = 2[c_3 J_{3,n} + c_2 J_{2,n} + c_1 J_{1,n} + c_0 J_{0,n}] \tag{37}$$

$$a_2 K_{2,n} + a_1 K_{1,n} + a_0 K_{0,n} = c_3 L_{3,n} + c_2 L_{2,n} + c_1 L_{1,n} + c_0 L_{0,n}. \tag{38}$$

Taking into account the different values of $I_{k,n}$, $J_{k,n}$, $K_{k,n}$, and $L_{k,n}$ given above, we obtain the *Laguerre–Freud's equations for the class* 1,

$$\begin{aligned} a_2 \gamma_1 &= -\psi(\beta_0) \\ (a_2 - 2nc_3)(\gamma_{n+1} + \gamma_n) &= 4c_3 \sum_{k=1}^{n-1} \gamma_k + 2 \sum_{k=0}^{n-1} \theta_{\beta_n}\phi(\beta_k) - \psi(\beta_n) \qquad n \geqslant 1 \end{aligned} \tag{39}$$

and

$$[a_2-(2n+1)c_3]\,\gamma_{n+1}\beta_{n+1}$$
$$=\sum_{k=0}^{n}\phi(\beta_k)+c_3\left[2\gamma_{n+1}\left(n\beta_n+\sum_{k=0}^{n}\beta_k\right)+3\sum_{k=1}^{n}\gamma_k(\beta_k+\beta_{k+1})\right]$$
$$+c_2\left[(2n+1)\gamma_{n+1}+2\sum_{k=1}^{n}\gamma_k\right]-(a_2\beta_n+a_1)\,\gamma_{n+1};\qquad n\geqslant 0, \tag{40}$$

where

$$\theta_\alpha\phi(x)=\frac{\phi(x)-\phi(\alpha)}{x-\alpha}.$$

To check the validity of Eqs. (39) and (40), we set, respectively,

$c_3=0,\ c_2=1,\ c_1=0,\ c_0=-1,\qquad a_2=0,\ a_1=-(\alpha+\beta+2),\ a_0=\beta-\alpha$

$c_3=0,\ c_2=1,\ c_1=c_0=0,\qquad a_2=0,\ a_1=-2a,\ a_0=-2$

$c_3=c_2=0,\ c_1=1,\ c_0=0,\qquad a_2=0,\ a_1=1,\ a_0=-\alpha-1$

$c_3=c_2=c_1=0,\ c_0=1,\qquad a_2=0,\ a_1=2,\ a_0=0.$

Then we recover the Jacobi, Bessel, Laguerre, and Hermite recurrence coefficients respectively [5].

C. *Generalized Exponential Weight*

Let $\mathscr{L}$ be represented by the weight function

$$\rho(x)=\exp\{-Q(x)\}\qquad \text{over the real line,}$$

with

$$Q(x)=\sum_{k=0}^{2m}q_kx^k,\qquad q_{2m}>0,$$

then the moments $\mu_n:=\langle\mathscr{L},x^n\rangle$ satisfy the following recurrence relations

$$\sum_{k=1}^{2m}kq_k\mu_{n+k-1}-n\mu_{n-1}=0\qquad \text{for}\quad n\geqslant 0. \tag{41}$$

In obedience to (41), the linear functional $\mathscr{L}$ satisfies

$$Q'(x)\mathscr{L}+D[\mathscr{L}]=0. \tag{42}$$

In this case Eqs. (24) and (25) give

$$\sum_{k=1}^{2m} kq_k \langle x^{k-1}\mathscr{L}, P_n^2 \rangle = 2\langle \mathscr{L}, P_n P_n' \rangle = 0 \tag{43}$$

$$\sum_{k=1}^{2m} kq_k \langle x^{k-1}\mathscr{L}, P_{n+1} P_n \rangle = \langle \mathscr{L}, (P_{n+1}P_n)' \rangle$$
$$= (n+1)\langle \mathscr{L}, P_n^2 \rangle \tag{44}$$

which may be written as

$$\sum_{k=1}^{2m} kq_k I_{k-1,n} = 0, \qquad \sum_{k=1}^{2m} kq_k K_{k-1,n} = (n+1) I_{0,n}$$

or, according to Lemma 3.1

$$\sum_{k=1}^{2m} kq_k C_{n,n}^{k-1} = 0 \tag{45}$$

$$\gamma_{n+1} \sum_{k=1}^{2m} kq_k C_{n+1,n}^{k-1} = n+1. \tag{46}$$

As we can see the Laguerre–Freud's equations are obtained by a simple computation of two components of the vectors $\mathbf{V}_k$ for $1 \leqslant k \leqslant 2m-1$.

For clarity's sake, let $m=2$. We have to compute $C_{n,n}^k$ and $C_{n+1,n}^k$ for $1 \leqslant k \leqslant 3$. In the light of Lemma 2.1 or Eq. (17), we have

$$\begin{gathered} C_{n,n}^1 = \beta_n \quad (n \geqslant 0), \qquad C_{n,n}^2 = \gamma_n + \beta_n^2 + \gamma_{n+1} \quad (n \geqslant 1) \\ C_{n,n}^3 = \gamma_n(\beta_{n-1} + \beta_n) + \beta_n(\gamma_n + \beta_n^2 + \gamma_{n+1}) \\ + \gamma_{n+1}(\beta_n + \beta_{n+1}) \quad n \geqslant 1) \\ C_{0,0}^2 = \beta_0^2 + \gamma, \qquad C_{0,0}^3 = \beta_0(\beta_0^2 + \gamma_1) + \gamma_1(\beta_0 + \beta_1) \end{gathered} \tag{47}$$

$$\begin{gathered} C_{n+1,n}^1 = 1 \quad (n \geqslant 0), \qquad C_{n+1,n}^2 = \beta_n + \beta_{n+1} \quad (n \geqslant 0) \\ C_{n+1,n}^3 = \gamma_n + \beta_n(\beta_n + \beta_{n+1}) + \gamma_{n+1} + \beta_{n+1}^2 + \gamma_{n+2} \quad (n \geqslant 1) \\ C_{1,0}^3 = \beta_0(\beta_0 + \beta_1) + \gamma_1 + \beta_1^2 + \gamma_2. \end{gathered} \tag{48}$$

Therefore the Laguerre–Freud's equations take the following forms

$$\left\{\begin{aligned} & 4q_4(\gamma_n\beta_{n-1} + 2\gamma_n\beta_n + \beta_n^3 + 2\gamma_{n+1}\beta_n + \gamma_{n+1}\beta_{n+1}) \\ & \qquad + 3q_3(\gamma_n + \beta_n^2 + \gamma_{n+1}) + 2q_2\beta_n + q_1 = 0 \quad (n \geqslant 1) && (49) \\ & 4q_4(\beta_0^3 + 2\gamma_1\beta_0 + \gamma_1\beta_1 + 3q_3(\beta_0^2 + \gamma_1) + 2q_2\beta_0 + q_1 = 0 \\ & \gamma_{n+1}\{4q_4(\gamma_n + \beta_n^2 + \beta_n\beta_{n+1} + \gamma_{n+1} + \beta_{n+1}^2 + \gamma_{n+2}) \\ & \qquad + 3q_3(\beta_n + \beta_{n+1}) + 2q_2\} = n+1 \quad (n \geqslant 1) && (50) \\ & \gamma_1[4q_4(\beta_0^2 + \beta_0\beta_1 + \gamma_1 + \beta_1^2 + \gamma_2) + 3q_3(\beta_0 + \beta_1) + 2q_2] = 1. \end{aligned}\right.$$

These two equations agree with the ones of Magnus (Eqs. (1.22) and (1.21) respectively in [23]), of course, with the suitable change of notations

$$\gamma_n \to a_n^2, \qquad \beta_n \to b_n, \qquad \text{and} \qquad q_k \to c_{4-k} \qquad \text{for} \quad 0 \leqslant k \leqslant 4.$$

Moreover, if we set $q_0 = q_1 = q_2 = 0$, and $q_4 = 1$, we note that $\rho(x) = \exp\{-x^4\}$ is an even function on $\mathbb{R}$ and the corresponding orthogonal polynomials constitute a symmetric sequence, that is to say that $\beta_n = 0$ for $n \geqslant 0$. Then Eq. (49) is a trivial identity. Eq. (50) is reduced to

$$4\gamma_{n+1}(\gamma_n + \gamma_{n+1} + \gamma_{n+2}) = n+1 \quad (n \geqslant 1), \qquad 4\gamma_1(\gamma_1 + \gamma_2) = 1 \quad (n = 0). \tag{51}$$

This last equation has been the object of considerable study [22, 7, 14, 19, 25, 26, 32].

IV. Applications

In the sequel, we will show the increasing complexity of the relations satisfied by β_n and γ_n with respect to changes in the weight support. We start with the Hermite weight function and successively truncate the support of this function in order to generate new orthogonal sequences. These new sequences still remain semi-classical.

1. *Hermite Case*

$$\phi(x) = 1, \qquad \psi(x) = 2x.$$

$\mathscr{L}_1$ satisfies $2x\mathscr{L}_1 + D[\mathscr{L}_1] = 0$. Equations (39) and (40) are equivalent to

$$\begin{aligned} 2\beta_0 &= 0 \\ 2\beta_n &= 0, \qquad n \geqslant 1 \\ 1 - 2\gamma_1 &= 0 \\ n + 1 - 2\gamma_{n+1} &= 0, \qquad n \geqslant 0 \end{aligned}$$

which lead to

$$\begin{aligned} \beta_n &= 0, \qquad && n \geqslant 0 \\ \gamma_{n+1} &= (n+1)/2, \qquad && n \geqslant 0. \end{aligned} \tag{52}$$

2. *Maxwell Case*

$\mathscr{L}$ is the Maxwell's linear functional [29, 30], which satisfies (11), i.e.,

$$(2x^2 - 1)\mathscr{L}_2 + D[x\mathscr{L}_2] = 0.$$

According to $\{(39), (40)\}$ we have

$$\begin{aligned} 2\gamma_1 &= 1 - 2\beta_0^2 \\ 2\gamma_{n+1} &= 2n - 2\gamma_n - (2\beta_n^2 - 1), \qquad n \geqslant 1 \\ 2\gamma_{n+1}\beta_{n+1} &= \sum_{k=0}^{n} \beta_k - 2\gamma_{n+1}\beta_n, \qquad n \geqslant 0 \end{aligned} \tag{53}$$

with $\beta_0 = 1/\sqrt{\pi}$.

Equations (53) may be written as

$$\begin{aligned} 2\gamma_1 &= 1 - 2\beta_0^2 \\ 2\gamma_{n+1} &= 2n + 1 - 2\gamma_n - 2\beta_n^2, \qquad n \geqslant 0 \\ 2\gamma_{n+1}\beta_{n+1} &= \sum_{k=0}^{n} \left[\sum_{h=0}^{k} \beta_h - (2k + 1 - 2\beta_k^2)\beta_k \right], \qquad n \geqslant 0. \end{aligned}$$

Equation (11) may be written in the following form

$$x\{2x\mathscr{L}_2 + D[\mathscr{L}_2] = 0$$

or

$$2x\mathscr{L}_2 + D[\mathscr{L}_2] = \langle 2x\mathscr{L}_2, 1 \rangle \delta_0,$$

where δ_c is the Dirac mass at the point c. The left hand-side of the previous equation is the same as the functional equation satisfied by the Hermite linear functional. We will say that Maxwell linear functional is the truncated Hermite linear functional.

3. *Truncated Maxwell Case*

$\mathscr{L}_3$ is the linear functional, defined by (12), i.e.,

$$(2x^3 - 2x^2 - 2x + 1)\mathscr{L}_3 + D[x(x-1)\mathscr{L}_3] = 0,$$

$$(x-1)\{(2x^2 - 1)\mathscr{L}_3 + D[x\mathscr{L}_3]\} = 0$$

or

$$(2x^2 - 1)\mathscr{L}_3 + D[x\mathscr{L}_3] = \langle ((2x^2 - 1)\mathscr{L}_3, 1 \rangle \delta_1.$$

As we can see the left side of this equation is the same as the functional equation satisfied by Maxwell linear functional and the right side is a Dirac mass, then $\mathscr{L}_3$ can be viewed as the truncated Maxwell's linear functional.

Application of (24) and (25) give

$$2I_{3,n} - 2I_{2,n} - 2I_{1,n} + I_{0,n} = 2(J_{2,n} - J_{1,n}) \tag{54}$$

$$2K_{3,n} - 2K_{2,n} - 2K_{1,n} + K_{0,n} = L_{2,n} - L_{1,n}. \tag{55}$$

First we have to compute $I_{3,n}$ and $K_{3,n}$,

$$\begin{aligned} I_{3,n} &= [\gamma_{n+1}(\beta_{n+1}+2\beta_n)+\gamma_n(2\beta_n+\beta_{n-1})+\beta_n^3]I_{0,n}, \qquad n\geqslant 1\\ I_{3,0} &= [\gamma_1(\beta_1+2\beta_0)+\beta_0^3]I_{0,0} \end{aligned} \tag{56}$$

$$\begin{aligned} K_{3,n} &= \gamma_{n+1}(\gamma_{n+2}+\gamma_{n+1}+\gamma_n+\beta_{n+1}^2+\beta_{n+1}\beta_n+\beta_n^2)I_{0,n}, \qquad n\geqslant 1\\ K_{3,0} &= \gamma_1(\gamma_2+\gamma_1+\beta_1^2+\beta_1\beta_0+\beta_0^2)I_{0,0}. \end{aligned} \tag{57}$$

Let us substitute the previous expressions in (54) and (55), we obtain

$$\begin{aligned} 2\gamma_{n+1}(\beta_{n+1}+2\beta_n-1) &= 2\left[(n-1)\beta_n+\sum_{k=0}^{n}\beta_k-\gamma_n\right.\\ &\qquad \left.\times(2\beta_n+\beta_{n-1})-n\right]-\psi(\beta_n), \qquad n\geqslant 1\\ 2\gamma_{n+1}\gamma_{n+2} &= \sum_{k=0}^{n}\phi(\beta_k)+2\sum_{k=1}^{n}\gamma_k\\ &\quad +\gamma_{n+1}[2n+1-\theta_{\beta_n}\psi(\beta_{n+1})\\ &\quad -2(\gamma_{n+1}+\gamma_n)], \qquad n\geqslant 1 \end{aligned} \tag{58}$$

$$\begin{aligned} 2\gamma_1(\beta_1+2\beta_0-1) &= -\psi(\beta_0)\\ 2\gamma_1\gamma_2 &= \phi(\beta_0)+\gamma_1[1-\theta_{\beta_0}\psi(\beta_1)-2\gamma_1]. \end{aligned} \tag{59}$$

4. *Truncated Maxwell's Companion Case*

$\tilde{\mathscr{L}}_3$ is represented by the weight function $\tilde{\rho}_3(x)=e^{-x^2}$ over $]-\infty,0]$ $U[1,\infty[$. It is easy to see that the moments $\mu_n:=\int_{-\infty}^{0}x^n e^{-x^2}\,dx+\int_1^{+\infty}x^n e^{-x^2}\,dx$ satisfy the following recurrence relation

$$2\mu_{n+3}-2\mu_{n+2}-2\mu_{n+1}+\mu_n-n(\mu_{n+1}-\mu_n)=0. \tag{60}$$

Thus the functional equation satisfied by $\tilde{\mathscr{L}}_3$ reads as $(2x^3-2x^2-2x+1)\tilde{\mathscr{L}}_3+D[x(x-1)\tilde{\mathscr{L}}_3]=0$. We have to note that $\tilde{\mathscr{L}}_3$ satisfies the same functional equation as $\mathscr{L}_3$, that is to say that the recurrence coefficients satisfy the same Laguerre–Freud's equations given by (58) and (59). If we denote β_n and γ_n (respectively $\tilde{\beta}_n$ and $\tilde{\gamma}_n$) the three-term recurrence coefficients associated with $\mathscr{L}_3$ (respectively $\tilde{\mathscr{L}}_3$), then (β_n,γ_n) and $(\tilde{\beta}_n,\tilde{\gamma}_n)$ satisfy the system $\{(58),(59)\}$ with the initial values

$$\beta_0=\frac{1-e^{-1}}{\gamma(1/2;1)}, \qquad \gamma_1=\frac{\gamma(3/2;1)}{\gamma(1/2;1)}-\left(\frac{1-e^{-1}}{\gamma(1/2;1)}\right)^2$$

$$\tilde{\beta}_0=\frac{e^{-1}-1}{2\Gamma(1/2)-\gamma(1/2;1)}; \qquad \tilde{\gamma}_1=\frac{2\Gamma(3/2)-\gamma(3/2;1)}{2\Gamma(1/2)-\gamma(1/2;1)}-\left(\frac{e^{-1}-1}{2\Gamma(1/2)-\gamma(1/2;1)}\right)^2,$$

where $\gamma(a, z)$ is the incomplete gamma function defined by

$$\begin{aligned}\gamma(a, z) &= \int_0^z t^{a-1} e^{-t}\, dt \qquad \text{for} \quad \operatorname{Re}(a) > 0 \\ &= a^{-1} z^a {}_1F_1\left(\begin{matrix} a \\ a+1 \end{matrix} \middle| -z\right) \\ &= a^{-1} z^a e^{-z} {}_1F_1\left(\begin{matrix} 1 \\ a+1 \end{matrix} \middle| z\right) \qquad [20, \text{p. } 220].\end{aligned}$$

5. *A Freud Weight*

$\mathscr{L}_4$ is represented by $\rho_4(x) = \exp(-x^{2m})$ over $[-c, c]$; it is easy to show that $\mathscr{L}_4$ satisfies

$$2(mx^{2m+1} - mc^2 x^{2m-1} - x)\mathscr{L}_4 + D[(x^2 - c^2)\mathscr{L}_4] = 0.$$

Hence, in this case the Laguerre–Freud's equations read as

$$\begin{cases} 2\{mI_{2m+1,n} - mc^2 I_{2m-1,n} - I_{1,n}\} = J_{2,n} - c^2 J_{0,n} \\ 2\{mK_{2m+1,n} - mc^2 K_{2m-1,n} - K_{1,n}\} = L_{2,n} - c^2 L_{0,n} \end{cases}$$

or, using Lemmas 3.1 and 3.2 and Eqs. (34), (36)

$$\begin{cases} 2\{mC^{2m+1}_{n,n} - mc^2 C^{2m-1}_{n,n} - C^1_{n,n}\} = (n-1)\beta_n + \sum\limits_{k=0}^{n} \beta_k, & n \geqslant 0 \\ 2\gamma_{n+1}\{mC^{2m+1}_{n+1,n} - mc^2 C^{2m-1}_{n+1,n} - C^1_{n+1,n}\} & \\ \quad = (2n+1)\gamma_{n+1} + 2\sum\limits_{k=1}^{n} \gamma_k + \sum\limits_{k=0}^{n} \beta_k^2 - c^2(n+1), & n \geqslant 0. \end{cases}$$

But $\rho_4(x)$ is an even function over a symmetric interval, then $\mathscr{L}_4$ is a symmetric linear functional, hence $\beta_n = 0$ for $n \geqslant 0$ and Laguerre–Freud's equations are reduced to

$$\begin{cases} 2(mC^{2m-1}_{n,n} - mc^2 C^{2m-1}_{n,n} - C^1_{n,n}\} = 0 & (61) \\ 2\gamma_{n+1}\{mC^{2m+1}_{n+1,n} - mc^2 C^{2m-1}_{n+1,n} - C^1_{n+1,n}\} & \\ \quad = (2n+1)\gamma_{n+1} + 2\sum\limits_{k=1}^{n} \gamma_k - c^2(n+1). & (62) \end{cases}$$

For convenience sake, we set $m = 2$. Therefore, we have to compute $C^k_{n,n}$ and $C^k_{n+1,n}$ for $k = 5, 3$, and 1. $C^1_{n,n}$ and $C^3_{n,n}$ are given by (47), where we have to equate β_n to zero, thus we have $C^1_{n,n} = C^3_{n,n} = 0$, $n \geqslant 0$.

For $C^5_{n,n}$, we use (17) and obtain $C^5_{n,n} = 0$. Therefore Eq. (61) is a trivial identity. Now to compute $C^1_{n+1,n}$ and $C^3_{n+1,n}$, we refer to (48), where we set $\beta_n=0$. We obtain $C^1_{n+1,n}=1$, $n\geqslant 0$ and $C^3_{n+1,n}=\gamma_n+\gamma_{n+1}+\gamma_{n+2}$, $n\geqslant 1$, $C^3_{1,0}=\gamma_1+\gamma_2$. For $C^5_{n+1,n}$, we use (17) and obtain

$$\begin{aligned}C^5_{n+1,n}&=\gamma_n(\gamma_{n-1}+\gamma_n+\gamma_{n+1})+\gamma_{n+1}(\gamma_n+\gamma_{n+1}+\gamma_{n+2})\\&\quad+\gamma_{n+2}(\gamma_n+\gamma_{n+1}+\gamma_{n+2}+\gamma_{n+3}),\qquad n\geqslant 1\\C^5_{1,0}&=\gamma_1(\gamma_1+\gamma_2)+\gamma_2(\gamma_1+\gamma_2+\gamma_3).\end{aligned}$$

So, Eq. (62) becomes

$$\begin{aligned}&4\gamma_{n+1}\{\gamma_n(\gamma_{n-1}+\gamma_n+\gamma_{n+1})+\gamma_{n+1}(\gamma_n+\gamma_{n+1}+\gamma_{n+2})\\&\qquad+\gamma_{n+2}(\gamma_n+\gamma_{n+1}+\gamma_{n+2}+\gamma_{n+3})-c^2(\gamma_n+\gamma_{n+1}+\gamma_{n+2})\}\\&\quad=(2n+3)\,\gamma_{n+1}+2\sum_{k=1}^{n}\gamma_k-(n+1)c^2,\qquad n\geqslant 1\\&4\gamma_1\{\gamma_1(\gamma_1+\gamma_2)+\gamma_2(\gamma_1+\gamma_2+\gamma_3)-c^2(\gamma_1+\gamma_2)\}\\&\quad=3\gamma_1-c^2\end{aligned}$$

or

$$\begin{cases}4\gamma_{n+1}\{\gamma_n(\gamma_{n-1}+\gamma_n+\gamma_{n+1})+(\gamma_{n+1}-c^2)\\\qquad\times(\gamma_n+\gamma_{n+1}+\gamma_{n+2})+\gamma_{n+2}(\gamma_n+\gamma_{n+1}+\gamma_{n+2}+\gamma_{n+3})\}\\\quad=(2n+3)\,\gamma_{n+1}+2\sum_{k=1}^{n}\gamma_k-(n+1)c^2,\qquad n\geqslant 1 & (63)\\4\gamma_1\{(\gamma_1-c^2)(\gamma_1+\gamma_2)+\gamma_2(\gamma_1+\gamma_2+\gamma_3)\}=3\gamma_1-c^2. & (64)\end{cases}$$

As noted by A. P. Magnus Eqs. (63) and (64) contain, as a particular solution, a solution of Eq. (51).

Let us now take $\mathscr{L}_4$; the linear functional represented by $\tilde{\rho}_4(x)=\exp(-x^{2m})$ over $]-\infty;-c]\ U[c;+\infty[$. Then $\tilde{\mathscr{L}}_4$ satisfies

$$2(mx^{2m+1}-mc^2x^{2m-1}-x)\tilde{\mathscr{L}}_4+D[(x^2-c^2)\tilde{\mathscr{L}}_4]=0.$$

Therefore γ_n verify the Eqs. (61) and (62). If we set $m=2$, and we denote γ_n (respectively $\tilde{\gamma}_n$) the coefficients of the three-term recurrence relation associated with $\mathscr{L}_4$ (respectively $\tilde{\mathscr{L}}_4$), γ_n and $\tilde{\gamma}_n$ satisfy the same Eqs. (63) and (64) with the initial values,

$$\gamma_0=\frac{1}{2}\gamma\left(\frac{1}{4};c^4\right),\qquad \gamma_1=\frac{\gamma(\frac{3}{4};c^4)}{\gamma(\frac{1}{4};c^4)},\qquad \gamma_2=\frac{(\frac{5}{4};c^4)}{\gamma(\frac{3}{4};c^4)}-\frac{\gamma(\frac{3}{4};c^4)}{\gamma(\frac{1}{4};c^4)}$$

$$\tilde{\gamma}_0=\frac{1}{4}\left[\Gamma\left(\frac{1}{4}\right)-\gamma\left(\frac{1}{4};c^4\right)\right],\qquad \tilde{\gamma}_1=\frac{2\Gamma(\frac{3}{4})-\gamma(\frac{3}{4};c^4)}{\Gamma(\frac{1}{4})-\gamma(\frac{1}{4};c^4)},$$

$$\tilde{\gamma}_2 = \frac{\Gamma(\frac{5}{4}) - \gamma(\frac{5}{4}; c^4)}{\Gamma(\frac{3}{4}) - \gamma(\frac{3}{4}, c^4)} - 2\,\frac{\Gamma(\frac{3}{4}) - \gamma(\frac{3}{4}; c^4)}{\Gamma(\frac{1}{4}) - \gamma(\frac{1}{4}, c^4)}.$$

As we can see we exhibit two situations where Laguerre–Freud's equations have more than one solution.

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